Illinois Institute of Technology
Physics

M.Sc. Comprehensive and Ph.D. Qualifying Examination

PART II

Saturday, August 27, 2016
12:00 - 4:00 PM

General Instructions

1. Each problem is to be done on a separate booklet. Label the front of each book with the identifying code letter you picked, the part number of the exam, and the number of the problem only; for example: A-I.6. Do not write your name or IIT ID number on any material handed in for grading.

2. Any numerical data not specified in a problem should be found in the table of constants at the front of the exam.

3. DON’T PANIC: It is not expected that each student will completely solve every problem. However, it is advisable to do a thorough job on those problems that you do solve.
**Physical Constants**

Speed of light in vacuum \[ c = 2.998 \times 10^8 \text{ m/s} \]

Planck’s constant \[ h = 6.626 \times 10^{-34} \text{ J·s} \]

\[ h = h/2\pi \]

\[ = 1.055 \times 10^{-34} \text{ J·s} \]

\[ = 6.582 \times 10^{-16} \text{ eV·s} \]

Permeability constant \[ \mu_0 = 4\pi \times 10^{-7} \text{ N/A}^2 \]

Permittivity constant \[ \frac{1}{4\pi\epsilon_0} = 8.898 \times 10^9 \text{ N·m}^2/\text{C}^2 \]

Fine structure constant \[ \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \]

\[ = 7.30 \times 10^{-3} = \frac{1}{137} \]

Gravitational constant \[ G = 6.67 \times 10^{-11} \text{ m}^3/\text{s}^2/\text{kg} \]

Avogadro’s number \[ N_A = 6.023 \times 10^{23} \text{ mole}^{-1} \]

Boltzmann’s constant \[ k = 1.381 \times 10^{-23} \text{ J/K} \]

\[ = 8.617 \times 10^{-5} \text{ eV/K} \]

\[ kT \text{ at room temperature} \]

\[ k \cdot 300 \text{ K} = 0.0258 \text{ eV} \]

Universal gas constant \[ R = 8.314 \text{ J/mole-K} \]

Stefan-Boltzmann constant \[ \sigma = 5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4 \]

Electron charge magnitude \[ e = 1.602 \times 10^{-19} \text{ C} \]

Electron rest mass \[ m_e = 9.109 \times 10^{-31} \text{ kg} \]

\[ = 0.5110 \text{ MeV/c}^2 \]

Neutron rest mass \[ m_n = 1.675 \times 10^{-27} \text{ kg} \]

\[ = 939.6 \text{ MeV/c}^2 \]

Proton rest mass \[ m_p = 1.672 \times 10^{-27} \text{ kg} \]

\[ = 938.3 \text{ MeV/c}^2 \]

Deuteron rest mass \[ m_d = 3.343 \times 10^{-27} \text{ kg} \]

\[ = 1875.6 \text{ MeV/c}^2 \]

Atomic mass unit (C^{12} = 12) \[ u = 1.661 \times 10^{-27} \text{ kg} \]

\[ = 931.5 \text{ MeV/c}^2 \]

Mass of earth \[ M_E = 5.98 \times 10^{24} \text{ kg} \]

Radius of earth \[ R_E = 6.37 \times 10^6 \text{ m} \]

Mass of sun \[ M_S = 1.99 \times 10^{30} \text{ kg} \]

Radius of sun \[ R_S = 6.96 \times 10^8 \text{ m} \]

Gravitational acceleration at earth’s surface \[ g = 9.81 \text{ m/s}^2 \]

Atmospheric pressure \[ = 1.01 \times 10^5 \text{ N/m}^2 \]

Radius of earth’s orbit \[ = 1.50 \times 10^{11} \text{ m} \]

Radius of moon’s orbit \[ = 3.84 \times 10^8 \text{ m} \]

**Conversion Factors**

\[
1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \\
1 \text{ Å} = 10^{-10} \text{ m} \\
1 \text{ barn (b)} = 10^{-28} \text{ m}^2 \\
0^\circ \text{ Celsius} = 273.16 \text{ K} \\
1 \text{ J} = 6.242 \times 10^{18} \text{ eV} \\
1 \text{ Fermi} = 10^{-15} \text{ m} \\
1 \text{ in} = 2.54 \text{ cm} \\
1 \text{ cal} = 4.19 \text{ J}
\]
**Problem 1:** A conducting wire of length $L$, cross-sectional radius $R$, and resistivity $\rho$ is connected to a battery as shown. Assume the potential difference between two ends of the wire is $V$.

a) Determine the magnitude and direction of the electric $E$ and magnetic $B$ fields at the surface of the wire. *Hint:* Sketching some field lines is sufficient to indicate directions.

b) Show that $\int \mathbf{S} \cdot d\mathbf{A}$ is equivalent to the power dissipated in the wire as Joule heat and calculate it explicitly. $\mathbf{S}$ is the Poynting vector.

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**Problem 2:** A sphere or radius $r$ carries a surface charge of density $a \cdot r$, where $a$ is a constant vector, and $r$ is a radius-vector with the origin at the center of the sphere. Find an electric field $\mathbf{E}$ at the center of the sphere.

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**Problem 3:**
Solve Laplace’s equation in the rectangular region shown in the figure, i.e., for

$$\begin{align*}
0 \leq x \leq a, \\
0 \leq y \leq \infty.
\end{align*}$$

The boundary conditions are $V = 0$ at $x = 0$, $a$, and $V = V_0$ for $y = 0$ line segment.
Problem 4: Consider a one-dimensional harmonic oscillator

\[ \hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2. \]

a) Introduce annihilation and creation operators:

\[ a = \sqrt{\frac{m\omega}{2\hbar}} \hat{q} + i \sqrt{\frac{1}{2m\hbar\omega}} \hat{p}; \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \hat{q} - i \sqrt{\frac{1}{2m\hbar\omega}} \hat{p}. \]

b) Find their commutation relations and represent the Hamiltonian using these new operators.

c) Calculate the first-order correction to the lowest energy level caused by an anharmonic small perturbation

\[ \hat{H}_1 = \frac{1}{4}\lambda \hat{q}^4. \]

*Hint:* recall the definition of the ground state (vacuum): \( a |0\rangle = 0 \)

Problem 5: A particle of mass \( m \) is confined to a one-dimensional region \( 0 < x < L \). It moves freely in this region. The confining potential \( V(x) \) is zero when \( 0 < x < L \), and is infinite when \( x < 0 \) or \( x > L \).

a) Write down the eigenvalues and eigenfunctions of the Hamiltonian.

b) Now consider the modified system in which there is also a potential

\[ \delta V(x) = \frac{c x}{L}, \]

where the coefficient \( c \) is small relative to \( L \).

c) Compute the energy levels in the modified system, working to the first order in perturbation theory in \( c \).

Problem 6: Many of the properties of metals can be understood using the theory of the free electron gas.

a) Consider \( N \) electrons (fermions) confined to a box of side, \( L \). Starting with the Schrödinger equation and appropriate boundary conditions derive an expression for the Fermi velocity, \( v_F \) at \( T = 0 \).

b) Estimate the Fermi velocity for a typical monovalent metal such as Na or Cu, i.e. where each atom contributes one electron to the Fermi gas.
Problem 7: A charged pion at rest \((m_{\pi^-} = 140 \text{ MeV}/c^2)\) decays 99\% of the time to a muon \((m_\mu = 106 \text{ MeV}/c^2)\) and a massless neutrino.

(a) What is the speed of the outgoing muon?
(b) What is the speed of the outgoing neutrino?
(c) If the pion had been moving with an initial velocity \(\vec{v} = 0.9c \hat{x}\) in the LAB frame, and the muon comes out in the \(+\hat{x}\) direction, what is the muon’s velocity in the LAB frame?

Problem 8: A photon of a wavelength \(\lambda\) scatters off a free electron of mass \(m\) which is initially at rest. The scattered photon has a wavelength \(\lambda'\) and scattered at an angle \(\theta\) measured from the incident direction. Express you answers in term of \(\lambda\), \(m\), \(\theta\), and the Planck constant \(h\).

(a) Derive \(\lambda'\).
(b) Calculate a kinetic energy of the recoil electron.

Hint: Do not forget that the recoil electron might gain a lot of energy after the scattering.